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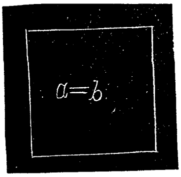
188. *All non-spaces of the same spaces are equal.*

Proof.—Suppose $|{}^\circ A$ and $|{}^\circ A_1$ to be two non-spaces of ${}^\circ A$. Then ${}^\circ A+|{}^\circ A=1={}^\circ A+|{}^\circ A_1$; whence by 185. remembering that ${}^\circ A.|{}^\circ A=0$ and ${}^\circ A.|{}^\circ A_1=0$, $|{}^\circ A=|{}^\circ A_1$.

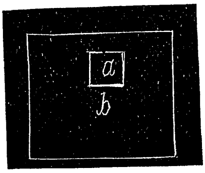
189. *The non-space of the non-space of a given space is the given space.*

Proof.—We have $|{}^\circ A+||{}^\circ A=1=|{}^\circ A+{}^\circ A$, and $||{}^\circ A.|{}^\circ A=0={}^\circ A\,|{}^\circ A$ (187). Whence, by 185, $||{}^\circ A={}^\circ A$.

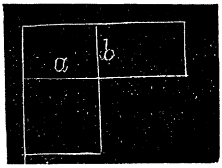
190. We give below four diagrams with descriptive names.



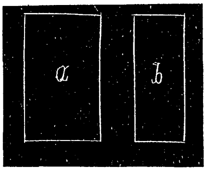
Identical Spaces.



Incident Space.



Cutting Spaces.



Disjunctive Spaces.

191. *Each of two spaces is incident to their sum. The product of two spaces is incident to each of them. Unity is the highest space, to which all other spaces are incident Nought is the lowest space, being included in all other spaces.* Compare the use of unity here with its use in Chapter V.

The theorems already given will suffice to show that the language and subject matter of the *Ausdehnungslehre* can be utilized in the study of logic. The material for this chapter is taken from Robert Grassmann's *Ausdehnungslehre* (Slettin, 1891). Robert Grassmann aided his brother Hermann in the preparation of the *Ausdehnungslehre* of 1862 and the *Formenlehre* of 1872, and was always deeply interested in the subject.

The article in the Britannica already referred to and Professor Stokes' article in the January (1900) number of The American Mathematical Monthly show that there is considerable diversity of opinion regarding the philosophy underlying the application of mathematics to logic.

* * *

# INTEGRATION OF ELLIPTIC INTEGRALS.

By G. B. M. ZEER, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

[Concluded from November Number.]

$$\int\frac{d\theta}{\cos^6\theta\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{1-e^2}\int\frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\cos^6\theta}$$

$$-\frac{e^2}{1-e^2}\int\frac{d\theta}{\cos^4\theta\sqrt{(1-e^2\sin^2\theta)}}=\frac{(8-19e^2+15e^4)}{15(1-e^2)^2}F(e,\ \theta)$$

$$-\frac{(8-23e^2+23e^4)}{15(1-e^2)^3}E(e,\ \theta)+\frac{(8-23e^2+23e^4)}{15(1-e^2)^3}\tan\theta\sqrt{(1-e^2\sin^2\theta)}$$

$$+\frac{\sin\theta}{5(1-e^2)\cos^5\theta}\sqrt{(1-e^2\sin^2\theta)}+\frac{4(1-2e^2)}{15(1-e^2)^2}\cdot\frac{\sin\theta}{\cos^3\theta}\sqrt{(1-e^2\sin^2\theta)}=S_3\ldots.(108).$$

From the foregoing we deduce the following relations :

$$V_2=e^2V_1+U_2,\quad V_3=e^2V_2+U_3.$$

$\therefore$ Generally, $V_n=e^2V_{n-1}+U_n$ ..........................................(109).

$3V_2=2(1+e^2)V_1-e^2V_0-(\cos/\sin^3\theta)\sqrt{(1-e^2\sin^2\theta)}$,

$5V_3=4(1+e^2)V_2-3e^2V_1-(\cos\theta/\sin^5\theta)\sqrt{(1-e^2\sin^2\theta)}$.

$\therefore$ Generally, $(2n-1)V_n=(2n-2)(1+e^2)V_{n-1}-(2n-3)e^2V_{n-2}$

$-(\cos\theta/\sin^{2n-1}\theta)\sqrt{(1-e^2\sin^2\theta)}$......................(110).

$$(1-e^2)S_2=R_2-e^2S_1,\quad (1-e^2)S_3=R_3-e^2S_2.$$

$\therefore$ Generally, $(1-e^2)S_n=R_n-e^2S_{n-1}$..............................(111).

$3(1-e^2)S_2=2(1-2e^2)S_1-e^2S_0+(\sin\theta/\cos^3\theta)\sqrt{(1-e^2\sin^2\theta)}$,

$5(1-e^2)S_3=4(1-2e^2)S_2-3e^2S_1+(\sin\theta/\cos^5\theta)\sqrt{(1-e^2\sin^2\theta)}$.

$\therefore$ Generally, $(2n-1)(1-e^2)S_n=(2n-2)(1-2e^2)S_{n-1}$

$-(2n-3)e^2S_{n-2}+(\sin\theta/\cos^{2n-1}\theta)\sqrt{(1-e^2\sin^2\theta)}$........(112).

$3U_2=(2-e^2)U_1-e^2(1-e^2)v_0$ - $(\cos\theta/\sin^3\theta)\sqrt{(1-e^2\sin^2\theta)}$,

$5U_3=(4-e^2)U_2-e^2(1-e^2)V_1-(\cos\theta/\sin^5)\sqrt{(1-e^2\sin^2\theta)}$.

$\therefore$ Generally, $(2n-1)U_n=(2n-2-e^2)u_{n-1}-e^2(1-e^2)v_{n-2}$

$-(\cos\theta/\sin^{2n-1}\theta)\sqrt{(1-e^2\sin^2)}$ .......................(113).

$3(1-e^2)R_2=(2-e^2)R_1-e^2S_0+(\sin\theta/\cos^3\theta)\sqrt{(1-e^2\sin^2\theta)}$,

$5(1-e^2)R_3=(4-3e^2)R_2-e^2S_1+(\sin\theta/\cos^5\theta)\sqrt{(1-e^2\sin^2\theta)}$.

$\therefore$ Generally, $(2n-1)(1-e^2)R_n=[2n(1-e^2)-(2-3e^2)]R_{n-1}$

$-e^2S_{n-2}+(\sin\theta/\cos^{2n-1}\theta)\sqrt{(1-e^2\sin^2\theta)}$ .............(114).

The following integrals come under the above .

$$\int\sqrt{(a^4+2b^2x^2+x^4)}dx,\quad \int\sqrt{(a^4-2b^2x^2+x^4)}dx,\ \text{where } a<b.$$

Let $b^2+(b^4-a^4)^{\frac{1}{2}}=m^2$, $b^2-(b^4-a^4)^{\frac{1}{2}}=n^2$.

$\therefore (a^4+2b^2x^2+x^4)=(x^2+m^2)(x^2+n^2)$.........................(115).

$(a^4-2b^2x^2+x^4)=(x^2-m^2)(x^2-n^2)$...........................(116).

In (115), let $x=n\tan\theta$. $\therefore dx=n\sec^2\theta d\theta$.

In (116), let $x=m\operatorname{cosec}\theta$. $\therefore dx=-m\operatorname{cosec}\theta\cot\theta d\theta$.

Let $e^2=(m^2-n^2)/m^2$, $k^2=n^2/m^2$.

$$\therefore \int\sqrt{(a^4+2b^2x^2+x^4)}dx=n^2\int\frac{\sqrt{[m^2-(m^2-n^2)\sin^2\theta]}d\theta}{\cos^4\theta}$$

$$=mn^2\int\frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\cos^4\theta}=mn^2R_2,\ \text{from (102).}$$

$$\int\sqrt{(a^4-2b^2x^2+x^4)}dx=-m^3\int\frac{\cos^2\theta\sqrt{(m^2-n^2\sin^2\theta)}d\theta}{\sin^4\theta}$$

$$=m^3\int\frac{\sqrt{(1-k^2\sin^2\theta)}d\theta}{\sin^2\theta}-m^3\int\frac{\sqrt{(1-k^2\sin^2\theta)}d\theta}{\sin^4\theta}=m^3(U_1-U_2),\ \text{from (97), (101)}$$

Let (3)$=G_1$, (5)$=G_2$, (10)$=G_3$, etc. (4)$=H_1$, (7)$=H_2$, (13)$=H_3$, etc. (8)$=I_1$, (15)$=I_2$, (22)$=I_3$, etc. (9)$=K_1$, (16)$=K_2$, (25)$=K_3$, etc.

and writing $(e,\theta)$ for $(e,\frac{1}{2}\pi)$ we get

$3e^2G_2=2(1+e^2)G_1-G_0$, $5e^2G_3=4(1+e^2)G_2-3G_1$.

$\therefore$ Generally, $(2n-1)e^2G_n=(2n-1)(1+e^2)G_{n-1}-(2n-3)G_{n-2}$ ....(117).

$3e^2H_2=(1-e^2)H_0-2(1-2e^2)H_1$, $5e^2H_3=3(1-e^2)H_1-4(1-2e^2)H_2$.

$\therefore$ Generally, $(2n-1)e^2H_n=(2n-2)(1-e^2)H_{n-2}$

$-(2n-1)(1-2e^2)H_{n-1}$..........(118).

$I_1=G_1-e^2G_2$, $I_2=G_2-e^2G_3$, $I_3=G_3-e^2G_4$.

∴ Generally, $I_n=G_n-e^2G_{n+1}$ ..............................(119).

$$K_1=(1-e^2)H_1+e^2H_2,\quad K_2=(1-e^2)H_2+e^2H_3,\quad K_3=(1-e^2)H_3+e^2H_4.$$

∴ Generally, $K_n=(1-e^2)H_n+e^2H_{n+1}$ ..........................(120).

$$\int\frac{d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}}=-\frac{\cot\theta}{\sqrt{(1-e^2\sin^2\theta)}}+e^2\int\frac{\cos^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$

$$\therefore\int\frac{\cos^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}=\frac{\cot\theta}{e^2\sqrt{(1-e^2\sin^2\theta)}}+\frac{1}{e^2}\int\frac{d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{e^2}F(e,\ \theta)-\frac{1}{e^2}E(e,\ \theta)+\frac{\sin\theta\cos\theta}{\sqrt{(1-e^2\sin^2\theta)}}=T_1\ ..(121).$$

$$\int\frac{d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}}=\frac{\tan\theta}{\sqrt{(1-e^2\sin^2\theta)}}-e^2\int\frac{\sin^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

$$\therefore\int\frac{\sin^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}=\frac{\tan\theta}{e^2\sqrt{(1-e^2\sin^2\theta)}}-\frac{1}{e^2}\int\frac{d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{e^2(1-e^2)}E(e,\ \theta)-\frac{1}{e^2}F(e,\ \theta)-\frac{\sin\theta\cos\theta}{(1-e^2)\sqrt{(1-e^2\sin^2\theta)}}=W_1\ ........(122).$$

$$\int\frac{d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}=T_1+W_1=\frac{1}{1-e^2}E(e,\ \theta)-\frac{e^2\sin\theta\cos\theta}{(1-e^2)\sqrt{(1-e^2\sin^2\theta)}}=W_0\ ..\ (123_0).$$

$$\int\frac{\cos^2\theta d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}}=-\frac{\cos^2\theta\cot\theta}{\sqrt{(1-e^2\sin^2\theta)}}-2(1-e^2)\int\frac{\cos^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$

$$-e^2\int\frac{\cos^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

$$\therefore\int\frac{\cos^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}=\frac{1}{e^2}\int\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}-\frac{1}{e^2}\int\frac{d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}}$$

$$-\frac{2(1-e^2)}{e^2}\int\frac{\cos^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}-\frac{\cos^2\theta\cot\theta}{e^2\sqrt{(1-e^2\sin^2\theta)}}=\frac{(2-e^2)}{e^4}E(e,\ \theta)$$

$$-\frac{2(1-e^2)}{e^4}F(e,\ \theta)-\frac{1-e^2}{e^2}.\frac{\sin\theta\cos\theta}{\sqrt{(1-e^2\sin^2\theta)}}=T_2\ ..\ (123).$$

$$\int\frac{\cos^4\theta d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}}=-\frac{\cot\theta\cos^4\theta}{\sqrt{(1-e^2\sin^2\theta)}}-4(1-e^2)\int\frac{\cos^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

$$-3e^2\int\frac{\cos^6\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

$$\therefore \int \frac{\cos^6\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}} = \frac{2}{3e^2}\int \frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}} - \frac{1}{3e^2}\int \frac{d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}}$$

$$-\frac{1}{3e^2}\int \frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}} - \frac{4(1-e^2)}{3e^2}\int \frac{\cos^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}} - \frac{\cot\theta\cos^4\theta}{3e^2\sqrt{(1-e^2\sin^2\theta)}}$$

$$= \frac{(8-17e^2+9e^4)}{3e^6}F(e,\ \theta) - \frac{(8-13e^2+3e^4)}{3e^6}E(e,\ \theta)$$

$$+\frac{(4-6e^2+3e^4)}{3e^4}\cdot\frac{\sin\theta\cos\theta}{\sqrt{(1-e^2\sin^2\theta)}} - \frac{\sin^3\theta\cos\theta}{3e^2\sqrt{(1-e^2\sin^2\theta)}} = T_3 \quad .. \ (124).$$

$$\int \frac{dx}{(a^4+2b^2x^2+x^4)^{\frac{3}{2}}} = \frac{1}{n^2}\int \frac{\cos^4\theta d\theta}{[m^2-(m^2-n^2)\sin^2\theta]^{\frac{3}{2}}} = \frac{1}{n^2 m^3}\int \frac{\cos^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$

$$= \frac{T_2}{n^2 m^3} \ldots . (125). \qquad \int \frac{dx}{(a^4-2b^2x^2+x^4)^{\frac{3}{2}}} = -\frac{1}{m^2}\int \frac{\sin^4\theta d\theta}{\cos^2\theta(m^2-n^2\sin^2\theta)^{\frac{3}{2}}}$$

$$= -\frac{1}{m^5}\int \frac{\sin^4\theta d\theta}{\cos^2\theta(1-k^2\sin^2\theta)^{\frac{3}{2}}} = \frac{2}{m^5}\int \frac{d\theta}{(1-k^2\sin^2\theta)^{\frac{3}{2}}}$$

$$-\frac{1}{m^5}\int \frac{d\theta}{\cos^2\theta(1-k^2\sin^2\theta)^{\frac{3}{2}}} - \frac{1}{m^5}\int \frac{\cos^2\theta d\theta}{(1-k^2\sin^2\theta)^{\frac{3}{2}}} = \frac{1}{m^5}(2W_0 - T_1)$$

$$-\frac{1}{m^5}\int \frac{d\theta}{\cos^2\theta(1-k^2\sin^2\theta)^{\frac{3}{2}}} = \frac{1}{m^5}(2W_0 - T_1) + \frac{k^2}{m^5(1-k^2)}\int \frac{d\theta}{(1-k^2\sin^2\theta)^{\frac{3}{2}}}$$

$$-\frac{1}{m^5(1-k^2)}\int \frac{d\theta}{\cos^2\theta\sqrt{(1-k^2\sin^2\theta)}} = \frac{1}{m^5}\left(\frac{2-k^2}{1-k^2}W_0 - \frac{1}{1-k^2}S_1 - T_1\right) \ .. \ (126).$$

In (126) and in the value of $\int \sqrt{(a^4-2b^2x^2+x^4)}dx$, $k$ is written instead of $e$.

$$\int \frac{\sin^2\theta d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}} = \frac{\tan\theta\sin^2\theta}{\sqrt{(1-e^2\sin^2\theta)}} - 2\int \frac{\sin^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$

$$+e^2\int \frac{\sin^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}. \qquad \therefore \int \frac{\sin^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}} = \frac{1}{e^2}\int \frac{d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}}$$

$$-\frac{1}{e^2}\int \frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}} + \frac{2}{e^2}\int \frac{\sin^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}} - \frac{\tan\theta\sin^2\theta}{e^2\sqrt{(1-e^2\sin^2\theta)}}$$

$$= \frac{2-e^2}{e^4(1-e^2)}E(e,\ \theta) - \frac{2}{e^4}F(e,\ \theta) - \frac{\sin\theta\cos\theta}{e^2(1-e^2)\sqrt{(1-e^2\sin^2\theta)}} = W_2 \ \ldots\ldots \ (127).$$

$$\int \frac{\sin^4\theta d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}} = \frac{\tan\theta\sin^4\theta}{\sqrt{(1-e^2\sin^2\theta)}} - 4\int \frac{\sin^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$

$$+3e^2\int \frac{\sin^6\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}. \qquad \therefore \int \frac{\sin^6\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}} = \frac{1}{3e^2}\int \frac{d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}}$$

$$-\frac{2}{3e^2}\int\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}+\frac{1}{3e^2}\int\frac{\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}+\frac{4}{3e^2}\int\frac{\sin^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$

$$-\frac{\tan\theta\sin^4\theta}{3e^2\sqrt{(1-e^2\sin^2\theta)}}=\frac{8-3e^2-2e^4}{3e^6(1-e^2)}E(e,\ \theta)-\frac{(8+e^2)}{3e^6}F(e,\ \theta)$$

$$-\frac{(1-e^2)\sin\theta\cos\theta}{3e^4\sqrt{(1-e^2\sin^2\theta)}}-\frac{\sin\theta\cos^3\theta}{3e^4\sqrt{(1-e^2\sin^2\theta)}}=W_3 \;\ldots\ldots\; (128).$$

Comparing (3) and (121) we get,

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{\cos^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

Comparing (6) and (123) we get,

$$3\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{\cos^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

Comparing (12) and (124) we get,

$$5\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^4\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{\cos^6\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

$\therefore$ Generally, $(2n-1)\displaystyle\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^{2n-2}\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{\cos^{2n}\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$

Comparing (4) and (122) we get,

$$\int_0^{\frac{1}{2}\pi}\frac{\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=(1-e^2)\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

Comparing (6) and (127) we get,

$$3\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=(1-e^2)\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

Comparing (11) and (128) we get,

$$5\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=(1-e^2)\int_0^{\frac{1}{2}\pi}\frac{\sin^6\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

$\therefore$ Generally, $(2n-1)\displaystyle\int_0^{\frac{1}{2}\pi}\frac{\sin^{2n-2}\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=(1-e^2)\int_0^{\frac{1}{2}\pi}\frac{\sin^{2n}\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$

$$\int\frac{d\theta}{\sin^2\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}}=\int\frac{d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}}+e^2\int\frac{d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$
$$=V_1+e^2W_0=L_1.$$

$$\int\frac{d\theta}{\cos^2\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}}=\frac{1}{1-e^2}\int\frac{d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}}-\frac{e^2}{1-e^2}\int\frac{d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$
$$=[1/(1-e^2)](S_1-e^2W_0)=O_2.$$

$$\int\frac{d\theta}{\sin^4\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}}=\int\frac{d\theta}{\sin^4\theta\sqrt{(1-e^2\sin^2\theta)}}+e^2\int\frac{d\theta}{\sin^2\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$

$$=V_2+e^2L_1=L_2.$$

$$\int\frac{d\theta}{\cos^4\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}}=\left(\int\frac{d\theta}{\cos^4\theta\sqrt{(1-e^2\sin^2\theta)}}-e^2O_1\right)\left(\frac{1}{1-e^2}\right)$$

$$=[1/(1-e^2)](S_2-e^2O_1)=O_2.$$

$\therefore$ Generally, $\displaystyle\int\frac{d\theta}{\sin^{2n}\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}}=V_n+e^2L_{n-1}=L_n.$

$$\int\frac{d\theta}{\cos^{2n}\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}}=\frac{1}{1-e^2}(S_n-e^2O_{n-1})=O_n.$$

The following general integrals may be of interest. The last term in the right hand member being the one sought.

$$\int\left(\frac{1}{\cos^2\theta}-\frac{1}{\sin^2\theta}\right)\frac{d\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2n-1)}}=\frac{\sec\theta\operatorname{cosec}\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2n-1)}}$$

$$+(2n-1)e^2\int\frac{d\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2n+1)}}.$$

$$\int\frac{\cos^{2m-2}\theta d\theta}{\sin^2\theta(1-e^2\sin^2\theta)^{\frac{1}{2}(2n-1)}}=-\frac{\cot\theta\cos^{2m-2}\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2n-1)}}$$

$$-(2m-2)(1-e^2)\int\frac{\cos^{2m-2}\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2n+1)}}-(2m-2n-1)e^2\int\frac{\cos^{2m}\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2n+1)}}.$$

$$\int\frac{\sin^{2m-2}\theta d\theta}{\cos^2\theta(1-e^2\sin^2\theta)^{\frac{1}{2}(2n-1)}}=\frac{\tan\theta\sin^{2m-2}\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2n-1)}}$$

$$-(2m-2)\int\frac{\sin^{2m-2}\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2n+1)}}+(2m-2n-1)e^2\int\frac{\sin^{2m}\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2n+1)}}.$$

$$\int\frac{d\theta}{\sin^{2m}\theta(1-e^2\sin^2\theta)^{\frac{1}{2}(2n-1)}}+e^2\int\frac{d\theta}{\sin^{2m-2}\theta(1-e^2\sin^2\theta)^{\frac{1}{2}(2n+1)}}$$

$$=\int\frac{d\theta}{\sin^{2m}\theta(1-e^2\sin^2\theta)^{\frac{1}{2}(2n+1)}}.$$

$$\frac{1}{1-e^2}\int\frac{d\theta}{\cos^{2m}\theta(1-e^2\sin^2\theta)^{\frac{1}{2}(2n-1)}}-\frac{e^2}{1-e^2}\int\frac{d\theta}{\cos^{2m-2}\theta(1-e^2\sin^2\theta)^{\frac{1}{2}(2n+1)}}$$

$$=\int\frac{d\theta}{\cos^{2m}\theta(1-e^2\sin^2\theta)^{\frac{1}{2}(2n+1)}}.$$